COMBINATORIAL PROOFS OF INFINITE VERSIONS OF THE HALES–JEWETT THEOREM

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Abstract. We provide new and purely combinatorial proofs of two infinite extensions of the Hales–Jewett theorem. The first one is due to T. Carlson and S. Simpson and the second one is due T. Carlson. Both concern infinite increasing sequences of finite alphabets.

1. introduction

The aim of this note is to provide purely combinatorial proofs of two known infinitary extensions of the Hales–Jewett theorem. To state the theorems we need first to recall the relevant terminology. By $\mathbb{N} = \{0, 1, ...\}$ we denote the set of all non negative integers. Let A be an *alphabet* (i.e. any non empty set). The elements of A will be called letters. By $W(A)$ we denote the set of *constant words over* A, that is all finite sequences with elements in A including the empty sequence. For $N \in \mathbb{N}$, by A^N, we denote all finite sequences from A, consisting of N letters. We also fix a variable $x \notin A$, then a *variable word over* A is an element in $W(A \cup \{x\}) \setminus W(A)$. The variable words will be denoted by $s(x), t(x), w(x)$, etc. A *left variable* word is a variable word such that x is its leftmost letter. Given a variable word $s(x)$ and $a \in A$, by $s(a)$ we denote the constant word in $W(A)$ resulting from the substitution of the variable x with the letter a. Given $q \in \mathbb{N}$ with $q \geq 1$, then a *q-coloring* of a set X is any map $c : X \to \{1, ..., q\}$. A subset Y of X will be called *monochromatic*, if there exists $1 \leq i \leq q$ such that $c(y) = i$, for all $y \in Y$. Finally, for every finite set X, by $|X|$ we denote its cardinality.

The following is the Hales–Jewett theorem [6].

Theorem 1. *For every positive integers p, q and every finite alphabet A with $|A| = p$, there exists a positive integer $N_0 = HJ(p, q)$ with the following property. For every $N \geq N_0$ and for every q-coloring of A^N there exists a variable word $w(x) \in (A \cup \{x\})^N \setminus A^N$ such that the set $\{w(a) : a \in A\}$ is monochromatic.*

The Hales–Jewett theorem gave birth to a whole new branch of research which concerns infinite extensions of it in the context of both finite and infinite alphabets (see [3], [4], [5], [8], [12], [13]). For an exposition of those results the reader can also refer to [11], [14].

The first theorem that we will prove is due to T. Carlson and S. Simpson [4].

Theorem 2. *Let $(A_n)_{n=0}^{\infty}$ be an increasing sequence of finite alphabets and let $A = \cup_{n\in\mathbb{N}} A_n$. Then for every finite coloring of $W(A)$ there exists a sequence $(w_n(x))_{n=0}^{\infty}$ of variable words over A such that for every $n \geq 1$, $w_n(x)$ is a left*

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variable word and for every $n \geq 0$ *the words of the form* $w_0(a_0)w_1(a_1)...w_n(a_n)$ *with* $a_0 \in A_0, a_1 \in A_1, ..., a_n \in A_n$ *are of the same color.*

The case where $A_n = A$ for all $n \in \mathbb{N}$ of the above theorem, was the key lemma for the main result of [4] and is commonly referred as the Carlson–Simpson theorem. As noted in [4, p. 274], the above reformulation in terms of an infinite increasing sequence of finite alphabets, is due to Miller and Prikry and it is closely related to an infinite extension of the Halpern–Läuchli theorem due to R. Laver [10].

The second theorem that we will prove is due to T. Carlson [3].

Theorem 3. *Let* $(A_n)_{n=0}^{\infty}$ *be an increasing sequence of finite alphabets and let* $A = \cup_{n\in\mathbb{N}} A_n$. *Then for every finite coloring of* $W(A)$ *there exists a sequence* $(w_n(x))_{n=0}^{\infty}$ *of variable words over* A *such that for every* $n \in \mathbb{N}$ *and every* $m_0 < m_1 < ... < m_n$, *the words of the form* $w_{m_0}(a_0)w_{m_1}(a_1)...w_{m_n}(a_n)$ *with* $a_0 \in A_{m_0}, a_1 \in A_{m_1}, ..., a_n \in A_{m_n}$ *are of the same color.*

Actually, Theorem 3 is a consequence of a more general result of T. Carlson (see [3, Theorem 15]). As shown in [8, §3], a left variable version of Theorem 3 is not true. However, such a version holds true for the case of a finite alphabet [12, Theorem 2.3].

The common approach of the proof of the above results is based on topological as well as algebraic notions of the Stone–Čech compactification of the related structures (see e.g. [9]). On the other hand, the proofs of Theorems 2 and 3 that we are going to present are based on the classical Hales–Jewett theorem and avoid the use of ultrafilters. Our approach has its origins in the proof of Hindman's theorem [7] due to J. E. Baumgartner [1]. Actually, a proof of a weaker version of Theorem 2 given in [11, §2.3] was the motivation for this note. The main difficulty that we encountered was the manipulation of the infinite sequence $(A_n)_{n=0}^{\infty}$ of alphabets. In particular, we remark that if one wishes to prove the aforementioned results for a finite alphabet then the proofs are considerably simpler.

2. Proof of theorem 2

2.1. **Preliminaries.** In this subsection we introduce some notation and terminology that we will use for the proof of Theorem 2. We fix an increasing sequence

$$A_0 \subseteq A_1 \subseteq ... \subseteq A_n \subseteq ...$$

of finite alphabets and we set

$$A = \cup_{n\in\mathbb{N}} A_n.$$

Let $V(A)$ be the set of all *variable words* (*over* A). By $V^{<\infty}(A)$ (resp. $V^{\infty}(A)$) we denote the set of all finite (resp. infinite) sequences of variable words. Also let $V^{\leq\infty}(A) = V^{<\infty}(A) \cup V^{\infty}(A)$. Generally, the elements of $V^{\leq\infty}(A)$ will be denoted by $\vec{s}, \vec{t}, \vec{w}$, etc.

2.1.1. *Reduced constant and variable span of a sequence of variable words.* Let $m \in \mathbb{N}$, $(s_n(x))_{n=0}^{m} \in V^{<\infty}(A)$ and $(k_n)_{n=0}^{m}$ be a strictly increasing finite sequence of non negative integers. The *reduced constant span of* $(s_n(x))_{n=0}^{m}$ *with respect to* $(A_{k_n})_{n=0}^{m}$ is defined to be the set

$$[(s_n(x))_{n=0}^{m} \parallel (A_{k_n})_{n=0}^{m}]_c = \{s_0(a_0)s_1(a_1)...s_m(a_m) : a_i \in A_{k_i} \text{ for all } 0 \leq i \leq m\}.$$

We also define the *reduced variable span of* $(s_n(x))_{n=0}^m$ *with respect to* $(A_{k_n})_{n=0}^m$ to be the set

$$[(s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m]_v = V(A) \cap [(s_n(x))_{n=0}^m \parallel (A_{k_n} \cup \{x\})_{n=0}^m]_c$$

that is the reduced variable span of $(s_n(x))_{n=0}^m$ with respect to $(A_{k_n})_{n=0}^m$ consists of all variable words of the form $s_0(b_0)...s_m(b_m)$ such that for each $0 \leq n \leq m$, $b_n \in A_{k_n} \cup \{x\}$ and for at least one n we have $b_n = x$.

Finally, for a non empty finite subset B of A we set

$$[(s_n(x))_{n=0}^m \parallel B]_c = \{s_0(b_0)s_1(b_1)...s_m(b_m) : b_i \in B \text{ for all } \ 0 \leq i \leq m\}$$

and

$$[(s_n(x))_{n=0}^m \parallel B]_v = V(A) \cap [(s_n(x))_{n=0}^m \parallel B \cup \{x\}]_c.$$

The above notation is naturally extended to infinite sequences of variable words as follows. Let $(s_n(x))_{n=0}^\infty \in V^\infty(A)$ and $(k_n)_{n=0}^\infty$ be a strictly increasing sequence of non negative integers. Then the reduced constant span of $(s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$, denoted by $[(s_n(x))_{n=0}^\infty \parallel (A_{k_n})_{n=0}^\infty]_c$ is the set

$$\begin{aligned}&\{s_0(a_0)s_1(a_1)...s_m(a_m) : m \in \mathbb{N}, a_i \in A_{k_i} \text{ for all } 0 \leq i \leq m\}\\ &= \cup_{m\in\mathbb{N}}[(s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m]_c\end{aligned}$$

and the reduced variable span of $(s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$ is the set

$$\begin{aligned}[(s_n(x))_{n=0}^\infty \parallel (A_{k_n})_{n=0}^\infty]_v &= V(A) \cap [(s_n(x))_{n=0}^\infty \parallel (A_{k_n} \cup \{x\})_{n=0}^\infty]_c\\ &= \cup_{m\in\mathbb{N}}[(s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m]_v.\end{aligned}$$

In the following we will also write $[\vec{s} \parallel (A_{k_n})_{n=0}^\infty]_c$ (resp. $[\vec{s} \parallel (A_{k_n})_{n=0}^\infty]_v$) to denote the the reduced constant (resp. variable) span of $\vec{s} = (s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$.

2.1.2. *Reduced k-block subsequences of a sequence of variable words.* We need to specify a notion of a "block subsequence" of a sequence of variable words. To this end we give the following definition.

Definition 4. *Let $k \in \mathbb{N}$.*

(i) *Let $l \in \mathbb{N}$, $\vec{t} = (t_n(x))_{n=0}^l \in V^{<\infty}(A)$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$. We say that $\vec{t}$ is a (finite) reduced k-block subsequence of $\vec{s}$ if there exist $0 = m_0 < ... < m_{l+1}$ such that*

$$t_i(x) \in [(s_n(x))_{n=m_i}^{m_{i+1}-1} \parallel (A_{k+n})_{n=m_i}^{m_{i+1}-1}]_v,$$

for all $0 \leq i \leq l$.

(ii) *Let $\vec{t} = (t_n(x))_{n=0}^\infty$, $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$. We say that $\vec{t}$ is a (infinite) reduced k-block subsequence of $\vec{s}$ if for every $l \in \mathbb{N}$ the finite sequence $(t_n(x))_{n=0}^l$ is a (finite) reduced k-block subsequence of $\vec{s}$.*

In the following we will write $\vec{t} \trianglelefteq_k \vec{s}$ whenever $\vec{t} \in V^{\leq\infty}(A)$, $\vec{s} \in V^\infty(A)$ and $\vec{t}$ is a reduced k-block subsequence of $\vec{s}$.

Taking into account that the sequence of alphabets $(A_n)_{n=0}^\infty$ is increasing, the next facts follow easily from the above definitions.

Fact 5. Let $k \in \mathbb{N}$ and $\vec{s}, \vec{t} \in V^\infty(A)$. If $\vec{t} \trianglelefteq_k \vec{s}$ then

$$[\vec{t} \parallel (A_{k+n})_{n=0}^\infty]_c \subseteq [\vec{s} \parallel (A_{k+n})_{n=0}^\infty]_c \text{ and } [\vec{t} \parallel (A_{k+n})_{n=0}^\infty]_v \subseteq [\vec{s} \parallel (A_{k+n})_{n=0}^\infty]_v.$$

Fact 6. Let $k_0, k_1 \in \mathbb{N}$, $\vec{s}, \vec{t}, \vec{w} \in V^\infty(A)$. If $k_0 \leq k_1$ and $\vec{w} \trianglelefteq_{k_0} \vec{t} \trianglelefteq_{k_1} \vec{s}$ then $\vec{w} \trianglelefteq_{k_1} \vec{s}$.

We close this section with a fusion lemma that we shall need later on.

Lemma 7. *Let $(\vec{t}_n)_{n=0}^\infty$ be a sequence in $V^\infty(A)$ and $(w_n(x))_{n=0}^\infty$ be a sequence of variable words. Also let $(k_n)_{n=0}^\infty$ and $(m_n)_{n=0}^\infty$ with $m_0 = 0$ be two sequences in $\mathbb{N}$. Let $\vec{t}_n = (t_i^{(n)}(x))_{i=0}^\infty$ for all $n \in \mathbb{N}$ and assume that for every $n \geq 1$ the following are satisfied.*

(i) $m_n \geq 1$ *and* $k_n = k_{n-1} + m_n$.
(ii) $w_{n-1}(x) \in [(t_i^{(n-1)}(x))_{i=0}^{m_n-1} \parallel (A_{k_{n-1}+i})_{i=0}^{m_n-1}]_v$.
(iii) $\vec{t}_n$ *is a reduced* k_n*-block subsequence of* $(t_i^{(n-1)}(x))_{i=m_n}^\infty$.

Then there exists a strictly increasing sequence $(p_n)_{n=0}^\infty$ in $\mathbb{N}$ with $p_0 = 0$ such that for every $n \in \mathbb{N}$ the following are satisfied.

(C1) $k_0 + p_n \geq k_n$.
(C2) $w_n(x) \in [(t_i^{(0)}(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k_0+i})_{i=p_n}^{p_{n+1}-1}]_v$.
(C3) $(w_i(x))_{i=n}^\infty$ *is a reduced* k_n*-block subsequence of* $\vec{t}_n$.

For the proof of Lemma 7, we shall need the following.

Sublemma 8. *Let $\vec{t} = (t_n(x))_{n=0}^\infty, \vec{t}\,' = (t'_n(x))_{n=0}^\infty \in V^\infty(A)$ and let k, k', p_1, p'_1 in $\mathbb{N}$ satisfying the following properties.*

(i) $k + p_1 \leq k' + p'_1$.
(ii) $(t_i(x))_{i=p_1}^\infty \trianglelefteq_{k'+p'_1} (t'_i(x))_{i=p'_1}^\infty$.

Let $p_2 \in \mathbb{N}$ with $p_2 > p_1$. Then there exists a $p'_2 \in \mathbb{N}$ with $p'_2 > p'_1$ such that the following hold.

(a) $p_2 - p_1 \leq p'_2 - p'_1$.
(b) $k + p_2 \leq k' + p'_2$.
(c) $(t_i(x))_{i=p_2}^\infty \trianglelefteq_{k'+p'_2} (t'_i(x))_{i=p'_2}^\infty$.
(d) $[(t_i(x))_{i=p_1}^{p_2-1} \parallel (A_{k+i})_{i=p_1}^{p_2-1}]_v \subseteq [(t'_i(x))_{i=p'_1}^{p'_2-1} \parallel (A_{k'+i})_{i=p'_1}^{p'_2-1}]_v$.

Proof. Using Definition 4 and assumption (ii) we find a unique positive integer d such that

$$t_{p_1}(x)...t_{p_2-1}(x) \in [(t'_i(x))_{i=p'_1}^{p'_1+d-1} \parallel (A_{k'+i})_{i=p'_1}^{p'_1+d-1}]_v \tag{1}$$

and

$$(t_i(x))_{i=p_2}^\infty \trianglelefteq_{k'+p'_1+d} (t'_i(x))_{i=p'_1+d}^\infty. \tag{2}$$

By (1) we easily conclude that

$$p_2 - p_1 \leq d. \tag{3}$$

Moreover, since the sequence of the alphabets $(A_n)_n$ is increasing, by (1) and assumption (i), we get that

$$[(t_i(x))_{i=p_1}^{p_2-1} \parallel (A_{k+i})_{i=p_1}^{p_2-1}]_v \subseteq [(t'_i(x))_{i=p'_1}^{p'_1+d-1} \parallel (A_{k'+i})_{i=p'_1}^{p'_1+d-1}]_v. \tag{4}$$

We set $p'_2 = p'_1 + d$. Then (a) follows from (3). Also, using (i), we have that

$$k + p_2 = k + p_1 + (p_2 - p_1) \leq k' + p'_1 + d = k' + p'_2$$

and so (b) is satisfied too. Finally, using (2) and (4) we easily derive (c) and (d) respectively. $\square$

Proof of Lemma 7. By induction for every $n \geq 1$ we select a finite sequence $(p_n^l)_{l=0}^n$ in $\mathbb{N}$ with $p_n^n = 0$ and $p_n^{n-1} = m_n$ satisfying for every $n \geq 1$ the following conditions.

(a) $p_{n-1}^l < p_n^l$ for all $0 \leq l \leq n-1$ and for every $1 \leq l \leq n-1$ we have
$$p_n^l - p_{n-1}^l \leq p_n^{l-1} - p_{n-1}^{l-1}.$$
(b) For every $1 \leq l \leq n$ we have
$$k_l + p_n^l \leq k_{l-1} + p_n^{l-1}.$$
(c) For every $1 \leq l \leq n$ we have
$$(t_i^{(l)}(x))_{i=p_n^l}^\infty \trianglelefteq_{k_{l-1}+p_n^{l-1}} (t_i^{(l-1)}(x))_{i=p_n^{l-1}}^\infty.$$
(d) For every $1 \leq l \leq n-1$ we have
$$[(t_i^{(l)}(x))_{i=p_{n-1}^l}^{p_n^l-1} \parallel (A_{k_l+i})_{i=p_{n-1}^l}^{p_n^l-1}]_v \subseteq [(t_i^{(l-1)}(x))_{i=p_{n-1}^{l-1}}^{p_n^{l-1}-1} \parallel (A_{k_{l-1}+i})_{i=p_{n-1}^{l-1}}^{p_n^{l-1}-1}]_v.$$

The above inductive construction is easily carried out using Sublemma 8.

We set $p_0 = 0$ and $p_n = p_n^0$, for all $n \geq 1$. We claim that the sequence $(p_n)_{n=0}^\infty$ is as desired. Indeed, by (a) we have $p_{n-1}^0 < p_n^0$ for all $n \geq 1$ and hence the sequence $(p_n)_{n=0}^\infty$ is strictly increasing. Moreover, by (b) we easily get (C1). Finally, by (ii) and (d) of Lemma 7, it follows that for every $n \in \mathbb{N}$ and every $0 \leq l \leq n$ we have

$$w_n(x) \in [(t_i^{(l)}(x))_{i=p_n^l}^{p_{n+1}^l-1} \parallel (A_{k_l+i})_{i=p_n^l}^{p_{n+1}^l-1}]_v. \tag{5}$$

Setting $l = 0$ in (5) we derive (C2). To verify (C3) we fix $n_0 \in \mathbb{N}$. Setting $l = n_0$ in (5) we get that $(w_n(x))_{n=n_0}^\infty$ is a reduced k_{n_0}-block subsequence of $\vec{t}_{n_0}$ and the proof of the lemma is complete. □

2.1.3. *Shifting sequences of variable words.* Let $s(x)$ be a variable word over A. By $s(x)^*$ we denote the maximal initial segment of $s(x)$ which is a constant word and by $s(x)^{**}$ the maximal final segment of $s(x)$ for which is a left variable word.

Clearly, $s(x) = s(x)^* s(x)^{**}$ for every $s(x) \in V(A)$. Moreover, if $s(x)$ is a left variable word then $s(x)^*$ is the empty word.

Definition 9. *We define a map $S : V^\infty(A) \to V^\infty(A)$ as follows. For every $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$ we set $S(\vec{s}) = (w_n(x))_{n=0}^\infty$, where $w_0(x) = s_0(x)s_1(x)^*$ and for every $n \geq 1$, $w_n(x) = s_n(x)^{**}s_{n+1}(x)^*$.*

Notice that for every $\vec{s} \in V^\infty(A)$, $S(\vec{s})$ is a sequence of variable words which all except perhaps the first, are left variable words. In addition, for every $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$ such that $s_n(x)$ is a left variable word for all $n \geq 1$, we have that $S(\vec{s}) = \vec{s}$.

It is easy to see that the map S preserves the infinite reduced k-block subsequences. More precisely, we have the following.

Fact 10. Let $k \in \mathbb{N}$ and $\vec{s}, \vec{t} \in V^\infty(A)$. If $\vec{t} \trianglelefteq_k \vec{s}$ then $S(\vec{t}) \trianglelefteq_k S(\vec{s})$.

2.1.4. *The notion of (S,k)-large families.* The next definition is crucial for the proof of Theorem 2.

Definition 11. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^\infty(A)$. Then E will be called (S,k)-large in $\vec{s}$ if*
$$E \cap [S(\vec{w}) \parallel (A_{k+n})_{n=0}^\infty]_c \neq \emptyset,$$
for every infinite reduced k-block subsequence $\vec{w}$ of $\vec{s}$.

In other words E is (S,k)-large in $\vec{s}$ if for every infinite reduced k-block subsequence $\vec{w}=(w_n(x))_{n=0}^\infty$ of $\vec{s}$ there exist $m\in\mathbb{N}$ and $a_i\in A_{k+i}$ for every $0\leq i\leq m$ such that

$$w_0(a_0)...w_m(a_m)w_{m+1}(x)^*\in E.$$

By Fact 6 we easily obtain the following.

Fact 12. Let $k\in\mathbb{N}$, $E\subseteq W(A)$ and $\vec{s}\in V^\infty(A)$ such that E is (S,k)-large in $\vec{s}$. Then for every infinite reduced k-block subsequence $\vec{t}$ of $\vec{s}$ we have that E is (S,k)-large in $\vec{t}$.

Lemma 13. *Let $k\in\mathbb{N}$, $E\subseteq W(A)$ and $\vec{s}\in V^\infty(A)$ such that E is (S,k)-large in $\vec{s}$. Let $r\geq 2$ and let $E=\bigcup_{i=1}^r E_i$. Then there exist $1\leq i\leq r$ and an infinite reduced k-block subsequence $\vec{t}$ of $\vec{s}$ such that E_i is (S,k)-large in $\vec{t}$.*

Proof. We first show the result for $r=2$. Assume to the contrary that for every infinite reduced k-block subsequence $\vec{t}$ of $\vec{s}$ neither E_1 nor E_2 is (S,k)-large in $\vec{t}$.

Then E_1 is not (S,k)-large in $\vec{s}$ and so there exists $\vec{t_1}\trianglelefteq_k\vec{s}$ such that

$$E_1\cap[S(\vec{t_1})\parallel(A_{k+n})_{n=0}^\infty]_c=\emptyset.$$

Similarly, E_2 is not (S,k)-large in $\vec{t_1}$ and so there exists $\vec{t_2}\trianglelefteq_k\vec{t_1}$ such that

$$E_2\cap[S(\vec{t_2})\parallel(A_{k+n})_{n=0}^\infty]_c=\emptyset.$$

Since $\vec{t_2}\trianglelefteq_k\vec{t_1}$, by Fact 6 we have that $\vec{t_2}\trianglelefteq_k\vec{s}$. Moreover, by Fact 10 we get that $S(\vec{t_2})\trianglelefteq_k S(\vec{t_1})$ and therefore by Fact 5 we have that

$$[S(\vec{t_2})\parallel(A_{k+n})_{n=0}^\infty]_c\subseteq[S(\vec{t_1})\parallel(A_{k+n})_{n=0}^\infty]_c.$$

Hence, $E\cap[S(\vec{t_2})\parallel(A_{k+n})_{n=0}^\infty]_c=\emptyset$, a contradiction since E is (S,k)-large in $\vec{s}$ and $t_2\trianglelefteq_k\vec{s}$. Hence, the lemma holds true for $r=2$. Using induction the result follows. □

2.2. **The main arguments.** We pass now to the core of the proof. We will need the next definition.

Definition 14. *Let E and F be non empty subsets of $W(A)$. We define*

$$E_F=\{z\in W(A): wz\in E \textit{ for every } w\in F\}.$$

Notice that for every $F_1,F_2\subseteq W(A)$ we have $(E_{F_1})_{F_2}=E_{F_1F_2}$, where $F_1F_2=\{w_1w_2:w_1\in F_1,w_2\in F_2\}$.

The next lemma is the first main step towards the proof of Theorem 2 and it is the point where we use the Hales–Jewett theorem.

Lemma 15. *Let $k\in\mathbb{N}$, $E\subseteq W(A)$ and $\vec{s}=(s_n(x))_{n=0}^\infty\in V^\infty(A)$ such that E is (S,k)-large in $\vec{s}$. Then there exist $m\geq 1$, $w(x)\in[(s_n(x))_{n=0}^{m-1}\parallel(A_{k+n})_{n=0}^{m-1}]_v$ and $\vec{t}\in V^\infty(A)$ with $\vec{t}\trianglelefteq_{k+m}(s_n(x))_{n=m}^\infty$ such that setting $F=\{w(a):a\in A_k\}$ then E_F is $(S,(k+m))$-large in $\vec{t}$.*

Proof. Fix $k\in\mathbb{N}$, $E\subseteq W(A)$ and $\vec{s}=(s_n(x))_{n=0}^\infty\in V^\infty(A)$ and assume that E is (S,k)-large in $\vec{s}$.

Claim 16. *There exist $r\in\mathbb{N}$ and a finite sequence $(w_i(x))_{i=0}^r$ with $(w_i(x))_{i=0}^r\trianglelefteq_k\vec{s}$ such that for every $w(x)\in V(A)$ with $(w_0(x),...,w_r(x),w(x))\trianglelefteq_k\vec{s}$ there exists $(a_0,...,a_r)\in\prod_{n=0}^r A_{k+n}$ such that $w_0(a_0)...w_r(a_r)w(x)^*\in E$.*

Proof of Claim 16. Assume that the conclusion fails. By induction we easily construct a sequence of variable words $\vec{w} = (w_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ with $w_0(x) = s_0(x)$, $\vec{w} \trianglelefteq_k \vec{s}$ and $[S(\vec{w}) \parallel (A_{k+n})_{n=0}^{\infty}]_c \subseteq E^c$, a contradiction. $\square$

Claim 17. *Let $(w_i(x))_{i=0}^{r}$ be as in Claim 16. Let $n_0 \geq 1$ be the unique integer such that $w_0(x)...w_r(x) \in [(s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1}]_v$ and set*

$$q = \prod_{n=0}^{r} |A_{k+n}|, \;\; N = HJ(|A_k|, q) \text{ and } m = n_0 + N,$$

where $HJ(|A_k|, q)$ is as in Theorem 1.

Then for every variable word $v(x) \in [(s_n(x))_{n=m}^{\infty} \parallel (A_{k+n})_{n=m}^{\infty}]_v$ there exist a constant word

$$w \in [(s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1}]_c$$

and a variable word

$$y(x) \in [(s_n(x))_{n=n_0}^{n_0+N-1} \parallel A_k]_v$$

such that $wy(b)v(x)^* \in E$ for all $b \in A_k$.

Proof of Claim 17. We set $B = A_k$ and we fix $v(x) \in [(s_n(x))_{n=m}^{\infty} \parallel (A_{k+n})_{n=m}^{\infty}]_v$. We will define a finite coloring of B^N, depending on $v(x)$, as follows. Let $\mathbf{b} = (b_0, ..., b_{N-1}) \in B^N$ be arbitrary and let $z(x) = s_{n_0}(b_0)...s_{n_0+N-1}(b_{N-1})v(x)$. Since $B = A_k \subseteq A_{k+n}$ for all $n \in \mathbb{N}$, we have that $z(x) \in [(s_n(x))_{n=n_0}^{\infty} \parallel (A_{k+n})_{n=n_0}^{\infty}]_v$. Therefore $(w_0(x), ..., w_r(x), z(x))$ is a finite reduced k-block subsequence of $\vec{s}$. Hence, by Claim 16 we may select $(a_0^{\mathbf{b}}, ..., a_r^{\mathbf{b}}) \in \prod_{n=0}^{r} A_{k+n}$ such that $w_0(a_0^{\mathbf{b}})...w_r(a_r^{\mathbf{b}})z(x)^* \in E$. We define

$$c_{v(x)} : B^N \to \prod_{n=0}^{r} A_{k+n}$$

by setting $c_{v(x)}(\mathbf{b}) = (a_0^{\mathbf{b}}, ..., a_r^{\mathbf{b}})$.

Since $q = \prod_{n=0}^{r} |A_{k+n}|$ and $N = HJ(|B|, q)$, by the Hales-Jewett Theorem and the way we define $c_{v(x)}$, we conclude that there exists a variable word

$$y(x) \in [(s_n(x))_{n=n_0}^{n_0+N-1} \parallel B]_v$$

and an $(r+1)$-tuple $(a_0, ..., a_r) \in \prod_{n=0}^{r} A_{k+n}$ such that

$$w_0(a_0)...w_r(a_r)y(b)v(x)^* \in E,$$

for all $b \in B$.

We set $w = w_0(a_0)...w_r(a_r)$. Since $w_0(x)...w_r(x) \in [(s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1}]_v$ we easily see that $w \in [(s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1}]_c$ and $wy(b)v(x)^* \in E$, for all $b \in B = A_k$. $\square$

Claim 18. *Let $(w_i(x))_{i=0}^{r}$ be as in Claim 16 and n_0, N and m be as in Claim 17. Let*

$$\mathbb{P} = [(w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r}]_c \times [(s_n(x))_{n=n_0}^{n_0+N-1} \parallel A_k]_v$$

and for every $(w, y(x)) \in \mathbb{P}$ let $F_{(w,y(x))} = \{wy(a) : a \in A_k\}$. Finally, let

$$E^* = \bigcup_{(w,y(x)) \in \mathbb{P}} E_{F_{(w,y(x))}},$$

where $E_{F_{(w,y(x))}} = \{z \in W(A) : uz \in E \text{ for all } u \in F_{(w,y(x))}\}$.

Then E^* is $(S, (k+m))$-large in $(s_n(x))_{n=m}^{\infty}$.

Proof of Claim 18. Let $\vec{t} = (t_n(x))_{n=0}^\infty$ be an arbitrary reduced $(k+m)$-block subsequence of $(s_n(x))_{n=m}^\infty$. Let $a_0 \in A_{k+m}$ and set $v(x) = t_0(a_0)t_1(x)$. Clearly,

$$v(x) \in [(s_n(x))_{n=m}^\infty \parallel (A_{k+n})_{n=m}^\infty]_v$$

and so, by Claim 17, there exists a pair $(w, y(x)) \in \mathbb{P}$ such that $wy(b)v(x)^* \in E$, for all $b \in A_k$ or equivalently $v(x)^* = t_0(a_0)t_1(x)^* \in E_{F_{(w,y(x))}} \subseteq E^*$. Therefore, E^* is $(S,(k+m))$-large in $(s_n(x))_{n=m}^\infty$. $\square$

We are now ready to finish the proof of the lemma. Indeed, by Claim 18 and Lemma 13, there exist a pair $(w_0, y_0(x)) \in \mathbb{P}$ and a reduced $(k+m)$-block subsequence $\vec{t}$ of $(s_n(x))_{n=m}^\infty$ such that $E_{F_{(w_0,y_0(x))}}$ is $(S,(k+m))$-large in $\vec{t}$. We set

$$w(x) = w_0y_0(x) \text{ and } F = F_{(w_0,y_0(x))} = \{w(a) : a \in A_k\}.$$

Then $w(x) \in [(s_n(x))_{n=0}^{m-1} \parallel (A_{k+n})_{n=0}^{m-1}]_v$ and E_F is $(S,(k+m))$-large in $\vec{t}$, as desired. $\square$

Lemma 19. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$ such that E is (S,k)-large in $\vec{s}$. Then there exist a sequence $(w_n(x))_{n=0}^\infty$ of variable words and two strictly increasing sequences $(k_n)_{n=0}^\infty$ and $(p_n)_{n=0}^\infty$ in $\mathbb{N}$, with $k_0 = k$ and $p_0 = 0$ such that setting for every $n \in \mathbb{N}$, $F_n = [(w_i(x))_{i=0}^n \parallel (A_{k_i})_{i=0}^n]_c$ then for every $n \in \mathbb{N}$ the following properties are satisfied.*

(P1) $k + p_n \geq k_n$.
(P2) $w_n(x) \in [(s_i(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k+i})_{i=p_n}^{p_{n+1}-1}]_v$.
(P3) E_{F_n} *is* (S, k_{n+1})*-large in* $(w_i(x))_{i=n+1}^\infty$.

Proof. By repeated use of Lemma 15 we obtain a sequence $(w_n(x))_{n=0}^\infty$ of variable words, a sequence $(\vec{t}_n)_{n=0}^\infty$ in $V^\infty(A)$ with $\vec{t}_0 = \vec{s}$, and two sequences $(k_n)_{n=0}^\infty$ and $(m_n)_{n=0}^\infty$ in $\mathbb{N}$, with $k_0 = k$ and $m_0 = 0$, such that setting for every $n \in \mathbb{N}$, $\vec{t}_n = (t_i^{(n)}(x))_{i=0}^\infty$ then for every $n \geq 1$ the following are satisfied.

(i) $m_n \geq 1$ and $k_n = k_{n-1} + m_n$.
(ii) $w_{n-1}(x) \in [(t_i^{(n-1)}(x))_{i=0}^{m_n-1} \parallel (A_{k_{n-1}+i})_{i=0}^{m_n-1}]_v$.
(iii) $\vec{t}_n$ is a reduced k_n-block subsequence of $(t_i^{(n-1)}(x))_{i=m_n}^\infty$.
(iv) The set $E_{F_{n-1}}$ is (S, k_n)-large in $\vec{t}_n$.

Assertions (i) - (iii) allow us to apply Lemma 7 and obtain a strictly increasing sequence $(p_n)_{n=0}^\infty$ with $p_0 = 0$ satisfying (C1)-(C3) of that lemma. Notice that (C1) and (C2) are identical to (P1) and (P2) respectively. By (C3) we have that $(w_i(x))_{i=n+1}^\infty$ is a reduced k_{n+1}-block subsequence of $\vec{t}_{n+1}$ and by (iv) above we have that E_{F_n} is (S, k_{n+1})-large in $\vec{t}_{n+1}$. Hence, by Fact 12 we have that (P3) is also satisfied. $\square$

Lemma 20. *Let $\vec{w} = (w_n(x))_{n=0}^\infty$, $(k_n)_{n=0}^\infty$ and $(p_n)_{n=0}^\infty$ be the sequences obtained in Lemma 19. Also let $F_n = [(w_i(x))_{i=0}^n \parallel (A_{k_i})_{i=0}^n]_c$, for all $n \in \mathbb{N}$. Then there exist a strictly increasing sequence $(r_n)_{n=0}^\infty$ in $\mathbb{N}$ with $r_0 = 0$ and a sequence $\vec{t} = (t_n(x))_{n=0}^\infty$ of variable words such that for every $n \geq 1$ the following are satisfied.*

(Q1) $t_n(x) \in [(w_{r_n+i}(x))_{i=0}^{r_{n+1}-r_n-1} \parallel (A_{k_{r_n}+i})_{i=0}^{r_{n+1}-r_n-1}]_v$.
(Q2) *For every* $u \in [(t_i(x))_{i=0}^{n-1} \parallel (A_{k_{r_{i+1}-1}})_{i=0}^{n-1}]_c$ *we have* $ut_n(x)^* \in E$.
(Q3) $[(t_i(x))_{i=0}^n \parallel (A_{k_{r_{i+1}-1}})_{i=0}^n]_c \subseteq F_{r_{n+1}-1}$.

Proof. We proceed by induction to the construction of the desired sequences. For $n=1$ we set $r_0=0$, $r_1=1$ and $t_0(x)=w_0(x)$. Let

$$G_0=[t_0(x)\parallel A_{k_{r_1-1}}]_c.$$

Then $G_0=F_0$ and therefore by Lemma 19 we have that E_{G_0} is (S,k_1)-large in $(w_i(x))_{i=1}^{\infty}$. Hence, there exist $r\in\mathbb{N}$ and $a_i\in A_{k_1+i}$, $0\le i\le r$ such that for every $u\in G_0$ we have

$$uw_{r_1}(a_0)...w_{r_1+r}(a_r)w_{r_1+r+1}(x)^*\in E.$$

We set $r_2=r_1+r+2$ and

$$t_1(x)=w_{r_1}(a_0)...w_{r_1+r}(a_r)w_{r_1+r+1}(x)=w_{r_1}(a_0)...w_{r_2-2}(a_r)w_{r_2-1}(x).$$

Notice that $t_1(x)^*=w_{r_1}(a_0)...w_{r_2-2}(a_r)w_{r_2-1}(x)^*$ and

$$[(t_i(x))_{i=0}^{1}\parallel(A_{k_{r_{i+1}-1}})_{i=0}^{1}]_c\subseteq[(w_i(x))_{i=0}^{r_2-1}\parallel(A_{k_i})_{i=0}^{r_2-1}]_c=F_{r_2-1}.$$

By the above we have that (Q1) - (Q3) are satisfied for $n=1$.

Assume that the construction has been carried out up to some $n\ge 1$. We set

$$G_n=[(t_i(x))_{i=0}^{n}\parallel(A_{k_{r_{i+1}-1}})_{i=0}^{n}]_c$$

and by our inductive hypothesis we have that $G_n\subseteq F_{r_{n+1}-1}$. Therefore by Lemma 19 we have that E_{G_n} is $(S,k_{r_{n+1}})$-large in $(w_i(x))_{i=r_{n+1}}^{\infty}$. Hence, there exist $r\in\mathbb{N}$ and $a_i\in A_{k_{r_{n+1}}+i}$, $0\le i\le r$ such that for every $u\in G_n$ we have

$$uw_{r_{n+1}}(a_0)...w_{r_{n+1}+r}(a_r)w_{r_{n+1}+r+1}(x)^*\in E.$$

We set $r_{n+2}=r_{n+1}+r+2$ and

$$\begin{aligned}t_{n+1}(x)&=w_{r_{n+1}}(a_0)...w_{r_{n+1}+r}(a_r)w_{r_{n+1}+r+1}(x)\\&=w_{r_{n+1}}(a_0)...w_{r_{n+2}-2}(a_r)w_{r_{n+2}-1}(x).\end{aligned}$$

Notice that $t_{n+1}(x)^*=w_{r_{n+1}}(a_0)...w_{r_{n+2}-2}(a_r)w_{r_{n+2}-1}(x)^*$ and

$$[(t_i(x))_{i=0}^{n+1}\parallel(A_{k_{r_{i+1}-1}})_{i=0}^{n+1}]_c\subseteq[(w_i(x))_{i=0}^{r_{n+2}-1}\parallel(A_{k_i})_{i=0}^{r_{n+2}-1}]_c=F_{r_{n+2}-1}.$$

It is easily checked that (Q1) - (Q3) are satisfied and the proof of the inductive step is complete. □

Corollary 21. *Let $k\in\mathbb{N}$, $E\subseteq W(A)$ and $\vec{s}\in V^{\infty}(A)$ such that E is (S,k)-large in $\vec{s}$. Then there exists a reduced k-block subsequence $\vec{t}$ of $\vec{s}$ such that*

$$[S(\vec{t})\parallel(A_{k+n})_{n=0}^{\infty}]_c\subseteq E.$$

Proof. Let $(r_n)_{n=0}^{\infty}$ and $\vec{t}=(t_n(x))_{n=0}^{\infty}$ be the sequences obtained in Lemma 20. Fix $j\in\mathbb{N}$. By (P2) of Lemma 19 we have that

$$w_{r_n+j}(x)\in[(s_i(x))_{i=p_{r_n+j}}^{p_{r_n+j+1}-1}\parallel(A_{k+i})_{i=p_{r_n+j}}^{p_{r_n+j+1}-1}]_v.$$

Moreover by (P1) of Lemma 19 we have that

$$k+p_{r_n+j}\ge k_{r_n+j}.$$

Hence, for all $j\in\mathbb{N}$,

$$[w_{r_n+j}(x)\parallel A_{k_{r_n+j}}]_c\subseteq[(s_i(x))_{i=p_{r_n+j}}^{p_{r_n+j+1}-1}\parallel(A_{k+i})_{i=p_{r_n+j}}^{p_{r_n+j+1}-1}]_c.$$

Therefore we obtain that

$$[(w_{r_n+j}(x))_{j=0}^{r_{n+1}-r_n-1}\parallel(A_{k_{r_n+j}})_{j=0}^{r_{n+1}-r_n-1}]_c\subseteq[(s_i(x))_{i=p_{r_n}}^{p_{r_{n+1}}-1}\parallel(A_{k+i})_{i=p_{r_n}}^{p_{r_{n+1}}-1}]_c$$

and so

$$[(w_{r_n+j}(x))_{j=0}^{r_{n+1}-r_n-1} \parallel (A_{k_{r_n}+j})_{j=0}^{r_{n+1}-r_n-1}]_v \subseteq [(s_i(x))_{i=p_{r_n}}^{p_{r_{n+1}}-1} \parallel (A_{k+i})_{i=p_{r_n}}^{p_{r_{n+1}}-1}]_v$$

Hence, by (Q1) of Lemma 20 we have that

$$t_n(x) \in [(s_i(x))_{i=p_{r_n}}^{p_{r_{n+1}}-1} \parallel (A_{k+i})_{i=p_{r_n}}^{p_{r_{n+1}}-1}]_v,$$

i.e. $\vec{t}$ is a reduced k-block subsequence of $\vec{s}$.

Recall that $S(\vec{t}) = (t_0(x)t_1(x)^*, t_1(x)^{**}t_2(x)^*, ...)$. We set $u_0(x) = t_0(x)t_1(x)^*$ and for all $n \geq 1$ we set $u_n(x) = t_n(x)^{**}t_{n+1}(x)^*$. Then

$$[S(\vec{t}) \parallel (A_{k+n})_{n=0}^{\infty}]_c = [(u_n(x))_{n=0}^{\infty} \parallel (A_{k+n})_{n=0}^{\infty}]_c.$$

Let $n \in \mathbb{N}$ and for every $0 \leq i \leq n$ let $a_i \in A_{k+i}$. Notice that

$$u_0(a_0)u_1(a_1)...u_n(a_n) = t_0(a_0)...t_n(a_n)t_{n+1}(x)^*.$$

We set $u = t_0(a_0)t_1(a_1)...t_n(a_n)$. Since the sequences $(k_n)_{n=0}^{\infty}$ and $(r_n)_{n=0}^{\infty}$ are increasing we obtain that

$$k_{r_{i+1}-1} \geq k_i \geq k + i.$$

Therefore,

$$u \in [(t_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n}]_c \subseteq [(t_i(x))_{i=0}^{n} \parallel (A_{k_{r_{i+1}-1}})_{i=0}^{n}]_c.$$

Hence by (Q2) of Lemma 20 we have that

$$ut_{n+1}(x)^* = t_0(a_0)...t_n(a_n)t_{n+1}(x)^* \in E.$$

Therefore $[S(\vec{t}) \parallel (A_{k+n})_{n=0}^{\infty}]_c \subseteq E$ and the proof is complete. □

Theorem 22. *Let $k \in \mathbb{N}$ and $\vec{v} = (v_n(x))_{n=0}^{\infty}$ be a sequence of variable words such that for every $n \geq 1$ $v_n(x)$ is a left variable word. Let $r \geq 2$ and $[\vec{v} \parallel (A_{k+n})_{n=0}^{\infty}]_c = \cup_{i=1}^{r} E_i$. Then there exist $1 \leq i \leq r$ and a reduced k-block subsequence $\vec{w} = (w_n(x))_{n=0}^{\infty}$ of $\vec{v}$ such that for every $n \geq 1$, $w_n(x)$ is a left variable word and $[\vec{w} \parallel (A_{k+n})_{n=0}^{\infty}]_c \subseteq E_i$.*

Proof. Since for every $n \geq 1$, $v_n(x)$ is a left variable word he have that $S(\vec{v}) = \vec{v}$ and so $[S(\vec{v}) \parallel (A_{k+n})_{n=0}^{\infty}]_c = [\vec{v} \parallel (A_{k+n})_{n=0}^{\infty}]_c = \cup_{i=1}^{r} E_i$. Let $\vec{t} \in V^{\infty}(A)$ such that $\vec{t} \trianglelefteq_k \vec{v}$. By Fact 10, we have that $S(\vec{t}) \trianglelefteq_k S(\vec{v})$ and by Fact 5 we obtain that $[S(\vec{t}) \parallel (A_{k+n})_{n=0}^{\infty}]_c \subseteq [S(\vec{v}) \parallel (A_{k+n})_{n=0}^{\infty}]_c$ and therefore $[S(\vec{t}) \parallel (A_{k+n})_{n=0}^{\infty}]_c \subseteq \cup_{i=1}^{r} E_i$. Thus, trivially, we get that $\cup_{i=1}^{r} E_i$ is (S,k)-large in $\vec{v}$. Hence, by Lemma 13, there exist $1 \leq i \leq r$ and $\vec{s} \trianglelefteq_k \vec{v}$ such that E_i is (S,k)-large in $\vec{s}$ and so by Corollary 21, there exists a $\vec{t} \trianglelefteq_k \vec{s}$ such that

$$[S(\vec{t}) \parallel (A_{k+n})_n]_c \subseteq E_i.$$

We set $\vec{w} = S(\vec{t})$. Since $\vec{t} \trianglelefteq_k \vec{s} \trianglelefteq_k \vec{v}$ we have $\vec{w} = S(\vec{t}) \trianglelefteq_k S(\vec{v}) = \vec{v}$. Hence, $\vec{w} \trianglelefteq_k \vec{v}$ and $[\vec{w} \parallel (A_{k+n})_n]_c \subseteq E_i$. The proof is complete. □

Proof of Theorem 2. Let $r \geq 2$ and let $W(A) = \cup_{i=1}^{r} E_i$. Let $\vec{v} = (x, x, ...)$. Then $[\vec{v} \parallel (A_n)_{n=0}^{\infty}]_c = W(A) = \cup_{i=1}^{r} E_i$. Applying Theorem 22 for $k = 0$, we obtain $1 \leq i \leq r$ and $\vec{w} = (w_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that for every $n \geq 1$, $w_n(x)$ is a left variable word and $[\vec{w} \parallel (A_n)_{n=0}^{\infty}]_c \subseteq E_i$. □

3. Proof of theorem 3

3.1. **Preliminaries.** As in Section 2.1 we fix for the following an increasing sequence

$$A_0 \subseteq A_1 \subseteq ... \subseteq A_n \subseteq ...$$

of finite alphabets and we set

$$A = \cup_{n\in\mathbb{N}} A_n.$$

Again, by $V(A)$ we denote the set of all variable words (over A). Also by $V^{<\infty}(A)$ (resp. $V^{\infty}(A)$) we denote the set of all finite (resp. infinite) sequences of variable words and let $V^{\leq\infty}(A) = V^{<\infty}(A) \cup V^{\infty}(A)$.

3.1.1. *Extracted constant and variable span of a sequence of variable words.* Let $m \in \mathbb{N}$, $(s_n(x))_{n=0}^m \in V^{<\infty}(A)$ and $(k_n)_{n=0}^m$ be a strictly increasing finite sequence of non negative integers. The *extracted constant span of* $(s_n(x))_{n=0}^m$ *with respect to* $(A_{k_n})_{n=0}^m$ denoted by $< (s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m >_c$ is defined to be the set

$$\bigcup_{n=0}^{m} \{s_{l_0}(a_0)...s_{l_n}(a_n) : 0 \leq l_0 < ... < l_n \leq m,\ a_i \in A_{k_{l_i}} \text{for all } 0 \leq i \leq n\}.$$

We also define the *extracted variable span of* $(s_n(x))_{n=0}^m$ *with respect to* $(A_{k_n})_{n=0}^m$ to be the set

$$< (s_n(x))_{n=0}^m \parallel (A_{k_n})_{n=0}^m >_v = V(A) \cap < (s_n(x))_{n=0}^m \parallel (A_{k_n} \cup \{x\})_{n=0}^m >_c,$$

that is the extracted variable span of $(s_n(x))_{n=0}^m$ with respect to $(A_{k_n})_{n=0}^m$ consists of all variable words of the form $s_{l_0}(b_0)s_{l_1}(b_1)...s_{l_n}(b_n)$ such that $0 \leq l_0 < l_1 < ... < l_n \leq m$, $b_i \in A_{k_{l_i}} \cup \{x\}$, for all $0 \leq i \leq n$ and for at least one i we have $b_i = x$.

The above notation extends to infinite sequences of variable words as follows. Let $(s_n(x))_{n=0}^\infty \in V^{\infty}(A)$ and $(k_n)_{n=0}^\infty$ be a strictly increasing sequence of non negative integers. Then the extracted constant span of $(s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$ denoted by $< (s_n(x))_{n=0}^\infty \parallel (A_{k_n})_{n=0}^\infty >_c$ is defined to be the set

$$\{s_{l_0}(a_0)s_{l_1}(a_1)...s_{l_n}(a_n) : n \in \mathbb{N},\ 0 \leq l_0 < l_1 < ... < l_n,\ a_i \in A_{k_{l_i}} \text{for all } 0 \leq i \leq n\}$$

and the extracted variable span of $(s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$ is the set

$$< (s_n(x))_{n=0}^\infty \parallel (A_{k_n})_{n=0}^\infty >_v = V(A) \cap < (s_n(x))_{n=0}^\infty \parallel (A_{k_n} \cup \{x\})_{n=0}^\infty >_c .$$

In the following we will also write $< \vec{s} \parallel (A_{k_n})_{n=0}^\infty >_c$ (resp. $< \vec{s} \parallel (A_{k_n})_{n=0}^\infty >_v$) to denote the the extracted constant (resp. variable) span of $\vec{s} = (s_n(x))_{n=0}^\infty$ with respect to $(A_{k_n})_{n=0}^\infty$.

3.1.2. *Extracted k-block subsequences of a sequence of variable words.* Next we specify a notion of a "block subsequence" of a sequence $\vec{s} \in V^{\infty}(A)$ which is related to the extracted span of $\vec{s}$.

Definition 23. *Let $k \in \mathbb{N}$.*

(i) *Let $l \in \mathbb{N}$, $\vec{t} = (t_n(x))_{n=0}^l \in V^{<\infty}(A)$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^{\infty}(A)$. We say that $\vec{t}$ is a (finite) extracted k-block subsequence of $\vec{s}$ if there exist $0 = m_0 < ... < m_{l+1}$ such that*

$$t_i(x) \in < (s_n(x))_{n=m_i}^{m_{i+1}-1} \parallel (A_{k+n})_{n=m_i}^{m_{i+1}-1} >_v,$$

for all $0 \leq i \leq l$.

(ii) *Let $\vec{t} = (t_n(x))_{n=0}^{\infty}, \vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$. We say that $\vec{t}$ is an (infinite) extracted k-block subsequence of $\vec{s}$ if for every $l \in \mathbb{N}$, the finite sequence $(t_n(x))_{n=0}^{l}$ is a (finite) extracted k-block subsequence of $\vec{s}$.*

In the following we will write $\vec{t} \leq_k \vec{s}$, whenever $\vec{t} \in V^{\leq\infty}(A)$, $\vec{s} \in V^{\infty}(A)$ and $\vec{t}$ is an extracted k-block subsequence of $\vec{s}$.

Notice that if $\vec{t}$ is a finite (resp. infinite) extracted k-block subsequence of $\vec{s}$ then by Definition 23 there exists a finite (resp. infinite) sequence of non negative integers (m_i) such that $t_i(x) \in < (s_n)_{n=m_i}^{m_{i+1}-1} \parallel (A_{k+n})_{n=m_i}^{m_{i+1}-1} >_v$. In contrast to the case of reduced k-block subsequences (see, Definition 4) this sequence of non negative integers is not necessarily unique. This is due to the way extracted spans and block subsequences are defined.

Moreover, taking into account that the sequence of alphabets $(A_n)_{n=0}^{\infty}$ is increasing, the next facts follow easily from the above definitions.

Fact 24. Let $k \in \mathbb{N}$ and $\vec{s}, \vec{t} \in V^{\infty}(A)$. If $\vec{t} \leq_k \vec{s}$ then

$$< \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq < \vec{s} \parallel (A_{k+n})_{n=0}^{\infty} >_c$$

and

$$< \vec{t} \parallel (A_{k+n})_{n=0}^{\infty} >_v \subseteq < \vec{s} \parallel (A_{k+n})_{n=0}^{\infty} >_v .$$

Fact 25. Let $k_0, k_1 \in \mathbb{N}$, $\vec{s}, \vec{t}, \vec{w} \in V^{\infty}(A)$. If $k_0 \leq k_1$ and $\vec{w} \leq_{k_0} \vec{t} \leq_{k_1} \vec{s}$ then $\vec{w} \leq_{k_1} \vec{s}$.

The following lemma corresponds to Lemma 7. The proof is similar.

Lemma 26. *Let $(\vec{t}_n)_{n=0}^{\infty}$ be a sequence in $V^{\infty}(A)$ and $(w_n(x))_{n=0}^{\infty}$ be a sequence of variable words. Also let $(k_n)_{n=0}^{\infty}$ and $(m_n)_{n=0}^{\infty}$ with $m_0 = 0$ be two sequences in $\mathbb{N}$. Let $\vec{t}_n = (t_i^{(n)}(x))_{i=0}^{\infty}$ for all $n \in \mathbb{N}$ and assume that for every $n \geq 1$ the following are satisfied.*

(i) $m_n \geq 1$ *and* $k_n = k_{n-1} + m_n$.

(ii) $w_{n-1}(x) \in < (t_i^{(n-1)}(x))_{i=0}^{m_n-1} \parallel (A_{k_{n-1}+i})_{i=0}^{m_n-1} >_v$.

(iii) $\vec{t}_n$ *is an extracted k_n-block subsequence of* $(t_i^{(n-1)}(x))_{i=m_n}^{\infty}$.

Then there exists a strictly increasing sequence $(p_n)_{n=0}^{\infty}$ in $\mathbb{N}$ with $p_0 = 0$ such that for every $n \in \mathbb{N}$ the following are satisfied.

(R1) $k_0 + p_n \geq k_n$.

(R2) $w_n(x) \in < (t_i^{(0)}(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k_0+i})_{i=p_n}^{p_{n+1}-1} >_v$.

(R3) $(w_i(x))_{i=n}^{\infty}$ *is a extracted k_n-block subsequence of* $\vec{t}_n$.

3.1.3. *The notion of k-large families.*

Definition 27. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$. Then E will be called k-large in $\vec{s}$ if*

$$E \cap < \vec{w} \parallel (A_{k+n})_{n=0}^{\infty} >_c \neq \emptyset,$$

for every infinite extracted k-block subsequence $\vec{w}$ of $\vec{s}$.

By Fact 25 we easily obtain the following.

Fact 28. Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then for every infinite extracted k-block subsequence $\vec{t}$ of $\vec{s}$ we have that E is k-large in $\vec{t}$.

Moreover, arguing as in Lemma 13 we obtain the lemma below.

Lemma 29. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^\infty(A)$ such that E is k-large in $\vec{s}$. Let $r \geq 2$ and let $E = \bigcup_{i=1}^r E_i$. Then there exist $1 \leq i \leq r$ and an infinite extracted k-block subsequence $\vec{t}$ of $\vec{s}$ such that E_i is k-large in $\vec{t}$.*

We will also need the following.

Lemma 30. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$ such that E is k-large in $\vec{s}$. Then for every $m \in \mathbb{N}$, E is $(k+m)$-large in $(s_n(x))_{n=m}^\infty$.*

Proof. Let $\vec{t} \in V^\infty(A)$ such that $\vec{t} \leq_{k+m} (s_n(x))_{n=m}^\infty$. It is easy to check that $\vec{t}$ is an extracted k-block subsequence of $\vec{s}$ and therefore, since E is k-large in $\vec{s}$, we obtain that

$$E \cap < \vec{t} \parallel (A_{k+n})_{n=0}^\infty >_c \neq \emptyset.$$

Moreover, since the sequence of alphabets (A_n) is increasing we get that

$$< \vec{t} \parallel (A_{k+n})_{n=0}^\infty >_c \subseteq < \vec{t} \parallel (A_{k+m+n})_{n=0}^\infty >_c .$$

Hence, $E \cap < \vec{t} \parallel (A_{k+m+n})_{n=0}^\infty >_c \neq \emptyset$ for every $\vec{t} \leq_{k+m} (s_n(x))_{n=m}^\infty$, i.e. E is $(k+m)$-large in $(s_n(x)_{n=m}^\infty$. □

3.2. **The main arguments.** We remind some notation from Section 2.1. For a non empty finite subset B of A we set

$$[(s_n(x))_{n=0}^m \parallel B]_c = \{s_0(b_0)s_1(b_1)...s_m(b_m) : b_i \in B \text{ for all } 0 \leq i \leq m\},$$

and

$$[(s_n(x))_{n=0}^m \parallel B]_v = V(A) \cap [(s_n(x))_{n=0}^m \parallel B \cup \{x\}]_c.$$

Lemma 31. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^\infty \in V^\infty(A)$ such that E is k-large in $\vec{s}$. Then there exist $m \in \mathbb{N}$ and $w(x) \in < (s_n(x))_{n=0}^m \parallel (A_{k+n})_{n=0}^m >_v$ such that $\{w(a) : a \in A_k\} \subseteq E$.*

Proof. Assume to the contrary that the conclusion fails. By induction we construct a sequence $\vec{w} = (w_n(x)) \leq_k \vec{s}$ such that $< \vec{w} \parallel (A_{k+n})_{n=0}^\infty >_c \subseteq E^c$ which is a contradiction since E is k-large in $\vec{s}$.

The general inductive step of the construction is as follows. Let $N \geq 1$ and assume that $(w_n(x))_{n=0}^{N-1}$ has been constructed so that

$$(w_n(x))_{n=0}^{N-1} \leq_k \vec{s} \text{ and } < (w_n(x))_{n=0}^{N-1} \parallel (A_{k+n})_{n=0}^{N-1} >_c \subseteq E^c.$$

Let $n_0 \geq 1$ be the least integer satisfying

$$w_0(x)...w_{N-1}(x) \in < (s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1} >_v$$

and let

$$q = 2^{\prod_{i=0}^{N-1}(|A_{k+i}|+1)}.$$

Let $H = HJ(|A_{k+N}|, q)$ be as in Theorem 1. For every $0 \leq i \leq N-1$ we set $\hat{A}_{k+i} = A_{k+i} \cup \{\emptyset\}$ and by convention $w_i(\emptyset) = \emptyset$. To each $w \in [(s_{n_0+n}(x))_{n=0}^{H-1} \parallel A_{k+N}]_c$ we assign the set of words

$$\{w_0(a_0)...w_{N-1}(a_{N-1})w : (a_0, ..., a_{N-1}) \in \prod_{i=0}^{N-1} \hat{A}_{k+i}\}.$$

Since $w_0(a_0)...w_{N-1}(a_{N-1})w$ belongs either to E or to E^c, the above correspondence induces a q-coloring on the set $[(s_{n_0+n}(x))_{n=0}^{N-1} \parallel A_{k+N}]_c$. Therefore, by the Hales–Jewett theorem there exists a variable word

$$w(x) \in [(s_{n_0+n}(x))_{n=0}^{N-1} \parallel A_{k+N}]_v$$

such that for every $(a_0, ..., a_{N-1}) \in \prod_{i=0}^{N-1} \hat{A}_{k+i}$ the set

$$\{w_0(a_0)...w_{N-1}(a_{N-1})w(a) : a \in A_{k+N}\}$$

either is included in E or is disjoint from E. By our initial assumption and since $A_k \subseteq A_{k+n}$, there is no $(a_0, ..., a_{N-1}) \in \prod_{i=0}^{N-1} \hat{A}_{k+i}$ satisfying the first alternative. So setting $w_N(x) = w(x)$ we easily see that $(w_n(x))_{n=0}^{N} \leq_k \vec{s}$ and

$$< (w_n(x))_{n=0}^{N} \parallel (A_{k+n})_{n=0}^{N} >_c \subseteq E^c.$$

The inductive step of the construction of $\vec{w}$ is complete and as we have already mentioned in the beginning of the proof this leads to a contradiction. □

The next lemma is crucial for the proof of Theorem 3 and it is the second and last point of the proof where the Hales–Jewett theorem is utilized.

Lemma 32. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then there exist $m \geq 1$, $w(x) \in< (s_n(x))_{n=0}^{m-1} \parallel (A_{k+n})_{n=0}^{m-1} >_v$ and $\vec{t} \in V^{\infty}(A)$ with $\vec{t} \leq_{k+m} (s_n(x))_{n=m}^{\infty}$ such that setting $F = \{w(a) : a \in A_k\}$ then $E \cap E_F$ is $(k+m)$-large in $\vec{t}$.*

Proof. Fix $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ and assume that E is k-large in $\vec{s}$.

Claim 33. *There exist $r \in \mathbb{N}$ and a finite sequence $(w_i(x))_{i=0}^{r}$ with $(w_i(x))_{i=0}^{r} \leq_k \vec{s}$ such that for every $w(x) \in V(A)$ with $(w_0(x), ..., w_r(x), w(x)) \leq_k \vec{s}$ there exist $v \in< (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c$ and $a \in A_{k+r+1}$ such that $v\, w(a) \in E$.*

Proof of Claim 33. Assume that the conclusion fails. By induction we easily construct a sequence of variable words $\vec{w} = (w_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ with $\vec{w} \leq_k \vec{s}$ and $w_0(x) = s_0(x)$ such that for every $n \in \mathbb{N}$, every $b \in< (w_i(x)_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} >_c$ and every $a \in A_{k+n+1}$ we have that $bw_{n+1}(a) \in E^c$. Setting for every $n \in \mathbb{N}$, $w_n'(x) = w_{2n}(x)w_{2n+1}(x)$ and $\vec{w}' = (w_n'(x))_{n=0}^{\infty}$ then we obtain that $\vec{w}' \leq_k \vec{s}$ and $< \vec{w}' \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq E^c$, a contradiction. □

Claim 34. *Let $(w_i(x))_{i=0}^{r}$ be as in Claim 33. Let $n_0 \geq 1$ be the least integer such that $w_0(x)...w_r(x) \in< (s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1} >_v$ and set*

$$q = \big| < (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c \big| \cdot |A_{k+r+1}|,$$

$$N = HJ(|A_k|, q) \text{ and } m = n_0 + N,$$

where $HJ(|A_k|, q)$ is as in Theorem 1.

Then for every variable word $v(x) \in< (s_n(x))_{n=m}^{\infty} \parallel (A_{k+n})_{n=m}^{\infty} >_v$ there exist a constant word

$$w \in< (s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1} >_c,$$

a letter $a \in A_{k+r+1}$ and a variable word

$$y(x) \in [(s_n(x))_{n=n_0}^{n_0+N-1} \parallel A_k]_v$$

such that $wy(b)v(a) \in E$ for all $b \in A_k$.

Proof of Claim 34. We set $B = A_k$ and we fix $v(x) \in< (s_n(x))_{n=m}^{\infty} \parallel (A_{k+n})_{n=m}^{\infty} >_v$. We will define a finite coloring of B^N, depending on $v(x)$, as follows. Let $\mathbf{b} = (b_0, ..., b_{N-1}) \in B^N$ be arbitrary and let $z(x) = s_{n_0}(b_0)...s_{n_0+N-1}(b_{N-1})v(x)$. Since $B = A_k \subseteq A_{k+n}$ for all $n \in \mathbb{N}$, we have that

$$z(x) \in< (s_n(x))_{n=n_0}^{\infty} \parallel (A_{k+n})_{n=n_0}^{\infty} >_v .$$

Therefore, $(w_0(x), ..., w_r(x), z(x))$ is a finite extracted k-block subsequence of $\vec{s}$. Hence, by Claim 33 there exists $w^{\mathbf{b}} \in< (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c$ and $a^{\mathbf{b}} \in A_{k+r+1}$ such that $w^{\mathbf{b}}z(a^{\mathbf{b}}) \in E$. We define

$$c_{v(x)} : B^N \to < (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c \times A_{k+r+1}$$

by setting

$$c_{v(x)}(\mathbf{b}) = (w^{\mathbf{b}}, a^{\mathbf{b}}).$$

Since $q = \Big| < (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c \Big| \cdot |A_{k+r+1}|$ and $N = HJ(|B|, q)$, by the Hales-Jewett Theorem and the way we define $c_{v(x)}$, we conclude that there exist a variable word

$$y(x) \in [(s_n(x))_{n=n_0}^{n_0+N-1} \parallel B]_v,$$

a constant word $w \in< (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c$ and $a \in A_{k+r+1}$ such that

$$wy(b)v(a) \in E,$$

for all $b \in B$. Since $w_0(x)...w_r(x) \in< (s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1} >_v$ we easily see that $w \in< (s_n(x))_{n=0}^{n_0-1} \parallel (A_{k+n})_{n=0}^{n_0-1} >_c$ and $wy(b)v(a) \in E$, for all $b \in B$. □

Claim 35. *Let $(w_i(x))_{i=0}^{r}$ be as in Claim 33 and n_0, N and m be as in Claim 34. Let*

$$\mathbb{P} =< (w_n(x))_{n=0}^{r} \parallel (A_{k+n})_{n=0}^{r} >_c \times [(s_n(x))_{n=n_0}^{n_0+N-1} \parallel A_k]_v$$

and for every $(w, y(x)) \in \mathbb{P}$ let $F_{(w,y(x))} = \{wy(a) : a \in A_k\}$. Finally, let

$$E^* = \bigcup_{(w,y(x))\in\mathbb{P}} E \cap E_{F_{(w,y(x))}},$$

where $E_{F_{(w,y(x))}} = \{z \in W(A) : uz \in E \text{ for all } u \in F_{(w,y(x))}\}$.

Then E^* is $(k+m)$-large in $(s_n(x))_{n=m}^{\infty}$.

Proof of Claim 35. Let $\vec{t} = (t_n(x))_{n=0}^{\infty}$ be an arbitrary extracted $(k+m)$-block subsequence of $(s_n(x))_{n=m}^{\infty}$. Since E is k-large in $\vec{s}$, by Lemma 30 we have that E is $(k+m)$-large in $(s_n(x))_{n=m}^{\infty}$ and hence E is $(k+m)$-large in $\vec{t}$. By Lemma 31 there exists a variable word $v(x)$ in $< (t_n(x))_{n=0}^{\infty} \parallel (A_{k+m+n})_{n=0}^{\infty} >_v$ such that for every $a \in A_{k+m}$, $v(a) \in E$. Since $v(x) \in< (t_n(x))_{n=0}^{\infty} \parallel (A_{k+m+n})_{n=0}^{\infty} >_v$ we easily obtain that $v(x) \in< (s_n(x))_{n=m}^{\infty} \parallel (A_{k+n})_{n=m}^{\infty} >_v$ and so by Claim 34 there exists a pair $(w, y(x)) \in \mathbb{P}$ and $a \in A_{k+r+1} \subseteq A_{k+m}$ such that $wy(b)v(a) \in E$, for all $b \in A_k$. Hence $v(a) \in E \cap E_{F_{(w,y(x))}} \subseteq E^*$. Moreover, $v(a) \in< (s_n(x))_{n=m}^{\infty} \parallel (A_{k+n})_{n=m}^{\infty} >_c$. Therefore, E^* is $(k+m)$-large in $(s_n(x))_{n=m}^{\infty}$. □

We are now ready to finish the proof of the lemma. Indeed, by Claim 35 and Lemma 29, there exist a pair $(w_0, y_0(x)) \in \mathbb{P}$ and an extracted $(k+m)$-block subsequence $\vec{t}$ of $(s_n(x))_{n=m}^{\infty}$ such that $E \cap E_{F_{(w_0,y_0(x))}}$ is $(k+m)$-large in $\vec{t}$. We set

$$w(x) = w_0y_0(x) \text{ and } F = F_{(w_0,y_0(x))} = \{w(a) : a \in A_k\}.$$

Then $w(x) \in< (s_n(x))_{n=0}^{m-1} \parallel (A_{k+n})_{n=0}^{m-1} >_v$ and $E \cap E_F$ is $(k+m)$-large in $\vec{t}$, as desired. □

Using Lemma 26 and arguing as in Lemma 19 we have the following.

Lemma 36. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} = (s_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then there exist a sequence $(w_n(x))_{n=0}^{\infty}$ of variable words and two strictly increasing sequences $(k_n)_{n=0}^{\infty}$ and $(p_n)_{n=0}^{\infty}$ in $\mathbb{N}$, with $k_0 = k$ and $p_0 = 0$ such that setting for every $n \in \mathbb{N}$, $F_n =< (w_i(x))_{i=0}^{n} \parallel (A_{k_i})_{i=0}^{n} >_c$ then for every $n \in \mathbb{N}$ the following properties are satisfied.*

(S1) *$k + p_n \geq k_n$.*
(S2) *$w_n(x) \in< (s_i(x))_{i=p_n}^{p_{n+1}-1} \parallel (A_{k+i})_{i=p_n}^{p_{n+1}-1} >_v$.*
(S3) *$E \cap E_{F_n}$ is k_{n+1}-large in $(w_i(x))_{i=n+1}^{\infty}$.*

Lemma 37. *Let $\vec{w} = (w_n(x))_{n=0}^{\infty}$, $(k_n)_{n=0}^{\infty}$ and $(p_n)_{n=0}^{\infty}$ be the sequences obtained in Lemma 36. Also let $F_n =< (w_i(x))_{i=0}^{n} \parallel (A_{k_i})_{i=0}^{n} >_c$, for all $n \in \mathbb{N}$. Then there exist a strictly increasing sequence $(r_n)_{n=0}^{\infty}$ in $\mathbb{N}$ with $r_0 = 0$ and a sequence $\vec{t} = (t_n(x))_{n=0}^{\infty}$ of variable words such that for every $n \geq 1$ the following are satisfied.*

(T1) *$t_n(x) \in< (w_{r_n+i}(x))_{i=0}^{r_{n+1}-r_n-1} \parallel (A_{k_{r_n}+i})_{i=0}^{r_{n+1}-r_n-1} >_v$.*
(T2) *For every $a \in A_{k_{r_n}}$ we have $t_n(a) \in E$ and $ut_n(a) \in E$, for every $u \in < (t_i(x))_{i=0}^{n-1} \parallel (A_{k_{r_i}})_{i=0}^{n-1} >_c$.*
(T3) *$< (t_i(x))_{i=0}^{n} \parallel (A_{k_{r_i}})_{i=0}^{n} >_c \subseteq F_{r_{n+1}-1}$.*

Proof. We proceed by induction to the construction of the desired sequences. For $n = 1$ we set $r_0 = 0$, $r_1 = 1$ and $t_0(x) = w_0(x)$. Let $G_0 =< t_0(x) \parallel A_{k_{r_0}} >_c$. Then $G_0 = F_0$ and therefore by Lemma 36 we have that $E \cap E_{G_0}$ is k_1-large in $(w_i(x))_{i=1}^{\infty}$. By Lemma 31 there exist $r \in \mathbb{N}$ and a variable word

$$t_1(x) \in< (w_{r_1+i}(x))_{i=0}^{r} \parallel (A_{k_{r_1}+i})_{i=0}^{r} >_v,$$

such that for every $a \in A_{k_{r_1}}$ and every $u \in< t_0(x) \parallel A_{k_{r_0}} >_c$, we have $t_1(a) \in E$ and $ut_1(a) \in E$. We set $r_2 = r_1 + r + 1$. It is straightforward that

$$t_1(x) \in< (w_{r_1+i}(x))_{i=0}^{r_2-r_1-1} \parallel (A_{k_{r_1}+i})_{i=0}^{r_2-r_1-1} >_c,$$

and

$$< (t_i(x))_{i=0}^{1} \parallel (A_{k_{r_i}})_{i=0}^{1} >_c \subseteq< (w_i(x))_{i=0}^{r_2-1} \parallel (A_{k_i})_{i=0}^{r_2-1} >_c = F_{r_2-1}.$$

By the above we have that (T1) - (T3) are satisfied for $n = 1$. Assume that the construction has been carried out up to some $n \geq 1$. We set

$$G_n =< (t_i(x))_{i=0}^{n} \parallel (A_{k_{r_i}})_{i=0}^{n} >_c$$

and by our inductive hypothesis we have that $G_n \subseteq F_{r_{n+1}-1}$. Therefore by Lemma 36 we have that $E \cap E_{G_n}$ is $k_{r_{n+1}}$-large in $(w_i(x))_{i=r_{n+1}}^{\infty}$. By Lemma 31 there exist $r \in \mathbb{N}$ and a variable word

$$t_{n+1}(x) \in< (w_{r_{n+1}+i}(x))_{i=0}^{r} \parallel (A_{k_{r_{n+1}}+i})_{i=0}^{r} >_v,$$

such that for every $a \in A_{k_{r_{n+1}}}$ and every $u \in< (t_i(x))_{i=0}^{n} \parallel (A_{k_{r_i}})_{i=0}^{n} >_c$, we have $t_{n+1}(a) \in E$ and $ut_{n+1}(a) \in E$. We set $r_{n+2} = r_{n+1} + r + 1$ and therefore

$$t_{n+1}(x) \in< (w_{r_{n+1}+i}(x))_{i=0}^{r_{n+2}-r_{n+1}-1} \parallel (A_{k_{r_{n+1}}+i})_{i=0}^{r_{n+2}-r_{n+1}-1} >_c,$$

and

$$< (t_i(x))_{i=0}^{n+1} \parallel (A_{k_{r_i}})_{i=0}^{n+1} >_c \subseteq< (w_i(x))_{i=0}^{r_{n+2}-1} \parallel (A_{k_i})_{i=0}^{r_{n+2}-1} >_c = F_{r_{n+2}-1}.$$

It is easy to check that (T1) - (T3) are satisfied and the proof is complete. $\square$

The following Corollary as well as its proof is similar to Corollary 21. For completeness we include its proof.

Corollary 38. *Let $k \in \mathbb{N}$, $E \subseteq W(A)$ and $\vec{s} \in V^{\infty}(A)$ such that E is k-large in $\vec{s}$. Then there exists an extracted k-block subsequence $\vec{u}$ of $\vec{s}$ such that*

$$< \vec{u} \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq E.$$

Proof. Let $(r_n)_{n=0}^{\infty}$ and $\vec{t} = (t_n(x))_{n=0}^{\infty}$ be the sequences obtained in Lemma 37. Using (S1) and (S2) of Lemma 36 and (T1) of Lemma 37 we have that

$$t_n(x) \in < (s_i(x))_{i=p_{r_n}}^{p_{r_{n+1}}-1} \parallel (A_{k+i})_{i=p_{r_n}}^{p_{r_{n+1}}-1} >_v,$$

i.e. $\vec{t}$ is an extracted k-block subsequence of $\vec{s}$.

For every $n \in \mathbb{N}$, We set $u_n(x) = t_{2n}(x)t_{2n+1}(x)$ and let $\vec{u} = (u_n(x))_{n=0}^{\infty}$. Clearly, $\vec{u}$ is an extracted k-block subsequence of $\vec{s}$. Since the sequences $(k_n)_{n=0}^{\infty}$ and $(r_n)_{n=0}^{\infty}$ are increasing we obtain that

$$< (t_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} >_c \subseteq < (t_i(x))_{i=0}^{n} \parallel (A_{k_{r_i}})_{i=0}^{n} >_c,$$

for all $n \in \mathbb{N}$. By the definition $\vec{u} = (u_n(x))_{n=0}^{\infty}$ we obtain that

$$< (u_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} >_c \subseteq < (t_i(x))_{i=0}^{2n+1} \parallel (A_{k+i})_{i=0}^{2n+1} >_c .$$

Hence,

$$< (u_i(x))_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} >_c \subseteq < (t_i(x))_{i=0}^{2n+1} \parallel (A_{k_{r_i}})_{i=0}^{2n+1} >_c,$$

for all $n \in \mathbb{N}$. By (T2) of Lemma 37 we have that for all $n \in \mathbb{N}$,

$$< (u_i(x)_{i=0}^{n} \parallel (A_{k+i})_{i=0}^{n} >_c \subseteq E,$$

and therefore $< \vec{u} \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq E$. $\square$

Theorem 39. *Let $k \in \mathbb{N}$ and $\vec{v} = (v_n(x))_{n=0}^{\infty}$ be a sequence of variable words. Let $r \geq 2$ and $< \vec{v} \parallel (A_{k+n})_{n=0}^{\infty} >_c = \cup_{i=1}^{r} E_i$. Then there exist $1 \leq i \leq r$ and an extracted k-block subsequence $\vec{u} = (u_n(x))_{n=0}^{\infty}$ of $\vec{v}$ such that $< \vec{u} \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq E_i$.*

Proof. Let $\vec{w} \in V^{\infty}(A)$ such that $\vec{w} \leq_k \vec{v}$. By Fact 24, we have that

$$< \vec{w} \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq < \vec{v} \parallel (A_{k+n})_{n=0}^{\infty} >_c$$

and therefore $< \vec{w} \parallel (A_{k+n})_{n=0}^{\infty} >_c \subseteq \cup_{i=1}^{r} E_i$. Thus, trivially, we get that $\cup_{i=1}^{r} E_i$ is k-large in $\vec{v}$. Hence, by Lemma 29, there exist $1 \leq i \leq r$ and $\vec{s} \leq_k \vec{v}$ such that E_i is k-large in $\vec{s}$ and so by Corollary 38, there exists a $\vec{u} \leq_k \vec{s}$ such that

$$< \vec{u} \parallel (A_{k+n})_n >_c \subseteq E_i.$$

Since $\vec{u} \leq_k \vec{s} \leq_k \vec{v}$, by Fact 25 we have that $\vec{u} \leq_k \vec{v}$ and the proof is complete. $\square$

Theorem 39 easily yields Theorem 3.

Proof of Theorem 3. Let $r \geq 2$ and let $W(A) = \cup_{i=1}^{r} E_i$. Let $\vec{v} = (x, x, ...)$. Then $< \vec{v} \parallel (A_n)_{n=0}^{\infty} >_c = W(A) = \cup_{i=1}^{r} E_i$. Applying Theorem 39 for $k = 0$, we obtain $1 \leq i \leq r$ and $\vec{w} = (w_n(x))_{n=0}^{\infty} \in V^{\infty}(A)$ such that $< \vec{w} \parallel (A_n)_{n=0}^{\infty} >_c \subseteq E_i$. $\square$

References

[1] J. E. Baumgartner, *A short proof of Hindman's theorem*, J. Combinatorial Theory (A) 17 (1974), 384–386.

[2] V. Bergelson, A. Blass, and N. Hindman, *Partition theorems for spaces of variable words*, Proc. London Math. Soc. 68 (1994), 449–476.

[3] T. Carlson, *Some unifying principles in Ramsey Theory*, Discrete Math. 68 (1988), 117–169.

[4] T. Carlson and S. Simpson *A dual form of Ramsey's Theorem*, Advances in Math. 53 (1984), 265–290.

[5] H. Furstenberg and Y. Katznelson, *Idempotents in compact semigroups and Ramsey Theory*, Israel J. Math. 68 (1989), 257–270.

[6] A. Hales and R. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. 106 (1963), 222–229.

[7] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbf{N}$, J. Combinatorial Theory (A) 17 (1974), 1–11.

[8] N. Hindman and R. McCutcheon, *Partition theorems for left and right variable words*, Combinatorica 24 (2004), no. 2, 271–286.

[9] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification*, Walter de Gruyter, Berlin, 1998.

[10] R. Laver, *Products of infinitely many perfect trees*, J. London Math. Soc. (2) 29 (1984), no. 3, 385-396.

[11] R. McCutcheon, *Elemental Methods in Ergodic Ramsey Theory*, Lecture Notes in Mathematics, Springer, 1999.

[12] R. McCutcheon, *Two new extensions of the Hales-Jewett theorem*, Electron. J. Combin. 7 (2000), Research Paper 49, 18 pp.

[13] H. J. Prömel and B. Voight, *Baire sets of k-parameter words are Ramsey*, Trans. Amer. Math. Soc. (1) 291 (1982), 189–201.

[14] S. Todorcevic, *Introduction to Ramsey Spaces*, Annals of Mathematics Studies, Princeton University Press, 2010.

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